Diracovo polje (jednačina)

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}}$$

Dirakove matrice $\{\gamma_\mu, \gamma^\nu\} = 2$

$$\gamma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}_{4\times4} \qquad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}_{4\times4}$$

$$\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \qquad \gamma_5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

$$\gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3$$

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu] \Rightarrow \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$

$$\not{a} = a^\mu\gamma_\mu = \cancel{\gamma^\mu a_\mu}\, a_\mu\gamma^\mu$$

$$\underline{(\gamma^0)^2 = 1} \qquad \underline{(\gamma^i)^2 = -1}$$

Reprezentacija γ matrica i γ^5 matrica

$$\gamma^5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}_*, \qquad \gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}_D \quad (tr\gamma^5 = 0)$$

$$\gamma^5 = -\frac{i}{4!} \epsilon^{\mu\nu\rho\sigma} \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma$$

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}_* \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}_*$$

* – hiralna reprezentacija

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}_D \qquad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}_D$$

D – Dirakova reprezentacija

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$
$$= i g^{00}\gamma_0 g^{11}\gamma_1 g^{22}\gamma_2 g^{33}\gamma_3$$
$$= i \underbrace{g^{00} g^{11} g^{22} g^{33}}_{(-1)} \gamma_0\gamma_1\gamma_2\gamma_3$$
$$= -i\gamma_0\gamma_1\gamma_2\gamma_3 = \gamma_5 \qquad (\gamma^5 = \gamma_5)$$

Hiralna reprezentacija ili Weyl-ova reprezentacija

$$\gamma^5 = -\frac{i}{4!} \epsilon^{\mu\nu\rho\sigma} \gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma =$$
$$= -\frac{i}{4!} \epsilon^{\mu\nu\rho\sigma} g_{\mu\mu'}\gamma^{\mu'} g_{\nu\nu'}\gamma^{\nu'} g_{\rho\rho'}\gamma^{\rho'} g_{\sigma\sigma'}\gamma^{\sigma'}$$
$$= -\frac{i}{4!} \epsilon_{\mu'\nu'\rho'\sigma'} \gamma^{\mu'}\gamma^{\nu'}\gamma^{\rho'}\gamma^{\sigma'}$$

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}_D \longrightarrow \text{Diracova reprezentacija}$$

$$\boxed{\not{p} = i\not{\partial}}$$

$$\not{p} = \gamma^\mu p_\mu = g_{\mu\nu}\gamma^\mu p^\nu = \gamma^0 p^0 - \vec{\gamma}\vec{p}$$

$$\boxed{\not{p} = \gamma^0 p_0 - \gamma^i p_i}$$

$$p_0 \equiv \frac{E}{c} = i\frac{\partial}{\partial x^0}\quad , \quad p_i = -i\frac{\partial}{\partial x^i} = -i\partial_i$$

$$\not{p} = i\gamma^0\frac{\partial}{\partial x^0} + i\gamma^i\partial_i \equiv i\gamma^0\partial_0 + i\gamma^i\partial_i$$

$$= i\gamma^\mu\partial_\mu = i\not{\partial}$$

$$\not{p} = i\not{\partial}$$

$$\boxed{\not{\partial} = \gamma^0\partial_0 + \gamma^i\partial_i}$$

1. Neka je data Dirakova j-na

$$i\hbar \frac{\partial \psi}{\partial t} = \left[-i\hbar c\, \vec{\alpha}\cdot\vec{\nabla} + \beta m c^2\right]\psi$$

gde su $\vec{\alpha}$ i β nepoznate. Da bi ova j-na opisivala slobodnu relativističnu česticu, potrebno je da zadovoljava i KG j-na. Na osnovu ovog ograničenja, naći $\vec{\alpha}$ i β.

Primeniti operator

$$i\partial_t = (-i\vec{\alpha}\cdot\vec{\nabla} + \beta m)$$ na levu i desnu stranu Dirakove j-ne. $(c = \hbar = 1)$

Sledi

$$-\frac{\partial^2\psi}{\partial t^2} = -\underbrace{\left(\alpha_i \frac{\partial}{\partial x_i}\right)\left(\alpha_j \frac{\partial}{\partial x_j}\right)\psi}_{\text{Objasniti}} - i m \{\alpha_i, \beta\}\frac{\partial \psi}{\partial x_i} +$$
$$+ m^2\beta^2\psi \qquad (1)$$

$$\boxed{-\alpha_i^2 \frac{\partial^2\psi}{\partial x_i^2} - (\alpha_i\alpha_j + \alpha_j\alpha_i)\frac{\partial^2\psi}{\partial x_i \partial x_j}}$$

KG j-na

$$(\partial^\mu\partial_\mu + m^2)\psi = 0$$

$$\frac{\partial^2\psi}{\partial t^2} - \Delta\psi + m^2\psi = 0$$

$$\frac{\partial^2\psi}{\partial t^2} = +\Delta\psi \cancel{+} m^2\psi \qquad (2)$$

$$\Downarrow$$

$$-\frac{\partial^2\psi}{\partial t^2} = -\Delta\psi + m^2\psi$$

Poređenjem (1) i (2)

$$\alpha_i \alpha_j + \alpha_j \alpha_i = 2\delta_{ij}$$

$$\{\alpha_i, \beta\} = 0 \qquad (3)$$

$$\beta^2 = 1$$

Ako su α_i i β brojevi, onda će prve dve relacije u (3) biti zadovoljene samo ako su bar 3 od 4 broja nule. A to se ne slaže sa $\alpha_i^2 = \beta^2 = 1$. Dakle, u pitanju su matrice!

Ako se uvede oznaka $\alpha_0 = \beta$, onda se (3) može zapisati kao

$$\{\alpha_\mu, \alpha_\nu\} = 2\delta_{\mu\nu}$$

Ove matrice moraju biti Ermitske.

Takođe, one su traga nula

$$\alpha_\mu \alpha_\nu = -\alpha_\nu \alpha_\mu$$

$$\alpha_\mu^2 \alpha_\nu = -\alpha_\mu \alpha_\nu \alpha_\mu$$

$$Tr(\alpha_\nu) = -Tr(\alpha_\mu \alpha_\nu \alpha_\mu) = -Tr(\alpha_\mu^2 \alpha_\nu) = -Tr(\alpha_\nu$$

Zato što je $\alpha_\mu^2 = 1$, svojstvene vrednosti su ± 1. Dakle α_μ su parne dimenzije (2×2, 4×4) npr.

U slučaju 2x2 matrica, imamo Paulijeve matrice koje zadovoljavaju $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$. Sa jediničnom matricom čine bazis u $\mathbb{C}^{22}$.
Ali, $\{\sigma_i, I\} = 2\sigma_i$ umesto nule. Dakle, mora se ići na 4x4 matrice. Sledeći izbor se pokazuje kao tačan:

$$\alpha_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}_{4\times 4}, \quad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}_{4\times 4}$$

~~FALI ZNAK?~~ NE!

Drugi zapis

$$\gamma^0 = \beta \qquad \gamma^i = \beta\alpha_i$$

$$i\hbar\gamma^0 \frac{\partial\psi}{\partial t} = (-i\hbar c\gamma^i \nabla_i + mc^2)\psi \,/ : c$$

$$i\gamma^0 \frac{\partial\psi}{\partial x^0} = (-i\gamma^i \frac{\partial}{\partial x^i} + m)\psi \quad ; \hbar = 1, c = 1$$

$$i\gamma^\mu \frac{\partial\psi}{\partial x^\mu} - m\psi = 0$$

v. pozadi

$$(i\gamma^\mu\partial_\mu - m)\psi$$

$$(i\not{\partial} - m)\psi = 0 \quad ; \quad \gamma^\mu A_\mu = \not{A}$$

$$\gamma^0 A^0 - \vec{\gamma}\vec{A} = \not{A}$$

$$i\hbar \frac{\partial\psi}{\partial t} = H_D\psi \qquad H_D = c\,\vec{\alpha}\cdot\vec{p} + \beta E_0$$

B. pozadi

2. Pokazati da iz $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$ sledi $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$. Koristiti def:

$\gamma_0 = \beta$ i $\gamma_i = \beta\alpha_i$, $\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$

Bitno za <u>nerelativistički limes</u>.

$$\{\gamma_i, \gamma_j\} = 2g_{ij} I = -2\delta_{ij} I$$

$$\{\beta\alpha_i, \beta\alpha_j\} = -2\delta_{ij} I$$

$$\beta\alpha_i\beta\alpha_j + \beta\alpha_j\beta\alpha_i = -2\delta_{ij} I$$

(pa se koristi antikomutator $\{\beta, \alpha_i\} = 0$)

$$-\alpha_i\alpha_j - \alpha_j\alpha_i = -2\delta_{ij} I$$

$$\boxed{\{\alpha_i, \alpha_j\} = 2\delta_{ij} I}$$

$$\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} = 2\delta_{ij} I$$

$$\begin{pmatrix} \sigma_i\sigma_j & 0 \\ 0 & \sigma_i\sigma_j \end{pmatrix} + \begin{pmatrix} \sigma_j\sigma_i & 0 \\ 0 & \sigma_j\sigma_i \end{pmatrix} = 2\delta_{ij} I$$

$$\begin{pmatrix} \{\sigma_i, \sigma_j\} & 0 \\ 0 & \{\sigma_i, \sigma_j\} \end{pmatrix} = \begin{pmatrix} 2\delta_{ij} & 0 \\ 0 & 2\delta_{ij} \end{pmatrix}$$

$$\Downarrow$$

$$\boxed{\{\sigma_i, \sigma_j\} = 2\delta_{ij}}$$

3. Neka je $U = e^{\beta \vec{\alpha} \cdot \vec{n}}$, gde su β i $\vec{\alpha}$ Dirakove matrice, $\vec{n}$ je jedinični vektor. Pokazati da važi:

$$\vec{\alpha}' = U \vec{\alpha} U^{+} = \vec{\alpha} - (1-U^2)(\vec{\alpha} \cdot \vec{n})\vec{n}$$

BHC f-la

$$e^{B} A e^{-B} = A + [B, A] + \frac{1}{2!}[B, [B, A]] + \dots$$

$$\boxed{\beta \alpha_i = -\alpha_i \beta \qquad \beta^{+} = \beta, \quad \alpha_i^{+} = \alpha_i}$$

$$U = e^{\beta \vec{\alpha} \cdot \vec{n}}, \quad U^{+} = ?$$

$$U^{+} = \left(e^{\beta \vec{\alpha} \cdot \vec{n}}\right)^{+} = \left(e^{\beta \alpha_i n_i}\right)^{+} = e^{(\beta \alpha_i)^{+} n_i}$$

$$= e^{\alpha_i^{+} \beta^{+} n_i} = e^{\alpha_i \beta n_i} = e^{-\beta \alpha_i n_i} = e^{-\beta \vec{\alpha} \cdot \vec{n}}$$

$$\boxed{U^{+} = e^{-\beta \vec{\alpha} \cdot \vec{n}}}$$

$$\vec{\alpha}' = e^{\beta \vec{\alpha} \cdot \vec{n}} \vec{\alpha} e^{-\beta \vec{\alpha} \cdot \vec{n}}$$

$$= \vec{\alpha} + [\beta \vec{\alpha} \cdot \vec{n}, \vec{\alpha}] + \frac{1}{2!}[\beta \vec{\alpha} \cdot \vec{n}, [\beta \vec{\alpha} \cdot \vec{n}, \vec{\alpha}]] + \dots$$

$$[\beta \vec{\alpha}\cdot\vec{n}, \vec{\alpha}] = [\beta \alpha_j n_j, \alpha_i] = n_j [\beta \alpha_j, \alpha_i]$$

$$= n_j (\beta \alpha_j \alpha_i - \alpha_i \beta \alpha_j) = n_j (\beta \alpha_j \alpha_i + \beta \alpha_i \alpha_j)$$

$$= n_j \beta \{\alpha_j, \alpha_i\} = 2\delta_{ij} n_j \beta = 2\beta n_i = 2\beta \vec{n}$$

$$[\beta \alpha_i n_i, 2\beta n_j] = 2 n_i n_j [\beta \alpha_i, \beta]$$

$$= 2 n_i n_j (\beta \alpha_i \beta - \beta^2 \alpha_i) = 2 n_i n_j (-\alpha_i - \alpha_i)$$

$$= -4 n_i n_j \alpha_i = -4 \vec{n} (\vec{\alpha}\cdot\vec{n})$$

$$[\beta \alpha_i n_i, -4 \underbrace{n_i n_j}_{\delta_{ij}} \alpha_i] = -4 n_i \delta_{ij} (\beta \alpha_i \alpha_i - \alpha_i \beta \alpha_i)$$

$$= -4 n_i \delta_{ij} (\beta \alpha_i^2 + \beta \alpha_i^2) = -8 n_i \delta_{ij} \beta = -8 \vec{n} \beta$$

$$[\beta \alpha_i n_i, -8 n_j \beta] = -8 n_i n_j (\beta \alpha_i \beta - \beta^2 \alpha_i)$$

$$= -8 n_i n_j (-\alpha_i - \alpha_i)$$

$$= 16 n_i n_j \alpha_i = 16 \vec{n} (\vec{\alpha}\cdot\vec{n})$$

$$\vec{\alpha}' = \vec{\alpha} + 2\beta \vec{n} - 2\vec{n}(\vec{\alpha}\cdot\vec{n}) - \frac{8}{3!} \vec{n} \beta + \frac{16}{4!} \vec{n} (\vec{\alpha}\cdot\vec{n})$$

Desna strana izraza

$$\vec{\alpha}\,' = \vec{\alpha} - (1-U^2)(\vec{\alpha}\cdot\vec{n})\vec{n} \quad ; \quad U = e^{\beta\vec{\alpha}\cdot\vec{n}}$$

$$U^2 = e^{2\beta\vec{\alpha}\cdot\vec{n}} = 1 + 2\beta\vec{\alpha}\cdot\vec{n} + \frac{(2\beta\vec{\alpha}\cdot\vec{n})^2}{2!} +$$

$$+ \frac{(2\beta\vec{\alpha}\cdot\vec{n})^3}{3!} + \frac{(2\beta\vec{\alpha}\cdot\vec{n})^4}{4!} + \dots$$

$$(\beta\vec{\alpha}\cdot\vec{n})^2 = \beta\alpha_i n_i \beta\alpha_j n_j = n_i n_j \beta\alpha_i \beta\alpha_j$$

$$= -n_i n_j \alpha_i \alpha_j = -\delta_{ij}\alpha_i\alpha_j = -\alpha_i^2 = -1$$

$$(\beta\vec{\alpha}\cdot\vec{n})^3 = (\beta\vec{\alpha}\cdot\vec{n})(\beta\vec{\alpha}\cdot\vec{n})^2 = -(\beta\cdot\vec{\alpha}\cdot\vec{n})$$

$$= -\beta\vec{\alpha}\vec{n}$$

$$(\beta\vec{\alpha}\cdot\vec{n})^4 = 1$$

$$U^2 = 1 + 2\beta\vec{\alpha}\cdot\vec{n} + \frac{4}{2}(-1) - \frac{8}{3!}\beta\vec{\alpha}\cdot\vec{n} + \frac{16}{4!} + \dots$$

$$U^2 = 1 + 2\beta\vec{\alpha}\cdot\vec{n} - 2 - \frac{8}{3!}\beta\vec{\alpha}\cdot\vec{n} + \frac{16}{4!} + \dots \Rightarrow$$

$$1 - U^2 = \cancel{1} - \cancel{1} - 2\beta\vec{\alpha}\cdot\vec{n} + 2 + \frac{8}{3!}\beta\vec{\alpha}\cdot\vec{n} - \frac{16}{4!} + \dots$$

$$-(1-U^2) = 2\beta\vec{\alpha}\cdot\vec{n} - 2 - \frac{8}{3!}\beta\vec{\alpha}\cdot\vec{n} + \frac{16}{4!}$$

$$\vec{L}' = \vec{L} + \left(2\beta\vec{L}\cdot\vec{n} - 2 - \frac{8}{3!}\beta\vec{L}\cdot\vec{n} + \frac{16}{4!} + \dots\right)(\vec{n}\cdot\vec{L})\vec{n}$$

$$\vec{L}' = \vec{L} + 2\beta(\vec{L}\cdot\vec{n})^2\vec{n} - 2(\vec{n}\cdot\vec{L})\vec{n} - \frac{8}{3!}\beta(\vec{n}\cdot\vec{L})^2\vec{n} + \frac{16}{4!}(\vec{n}\cdot\vec{L})\vec{n} + \dots$$

$$\vec{L}' = \vec{L} + 2\beta\vec{n} - 2\vec{n}(\vec{L}\cdot\vec{n}) - \frac{8}{3}\vec{n}\beta + \frac{16}{4!}(\vec{n}\cdot\vec{L})\vec{n} + \dots$$

što je jednako levoj strani izraza iz postavke.

$$\left[\ (\vec{n}\cdot\vec{L})^2 = n_i n_j L_i L_j = \delta_{ij} L_i L_j = L_i^2 = 1\ \right]$$

$\boxed{\text{Važno!}}$

4. Pokazati da je

1. $\gamma^{\mu +} = \gamma^0 \gamma^\mu \gamma^0$

2. $\sigma^{\mu\nu +} = \gamma^0 \sigma^{\mu\nu} \gamma^0$

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1. $(\gamma^0)^+ = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}^+ = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = \gamma^0 \gamma^0 \gamma^0 = \gamma^0$

$(\gamma^i)^+ = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}^+ = -\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} =$

$= -(\gamma^0)^2 \gamma^i = -\gamma^0 \gamma^0 \gamma^i = \gamma^0 \gamma^i \gamma^0$

$(\gamma^0)^2 = I \quad , \{\gamma^0, \gamma^i\} = 0$

2. $\sigma^{\mu\nu +} = -\frac{i}{2} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)^+$

$= -\frac{i}{2} (\gamma^{\nu +} \gamma^{\mu +} - \gamma^{\mu +} \gamma^{\nu +})$

$= -\frac{i}{2} (\gamma^0 \gamma^\nu \gamma^0 \gamma^0 \gamma^\mu \gamma^0 -$
$- \gamma^0 \gamma^\mu \gamma^0 \gamma^0 \gamma^\nu \gamma^0)$

$= -\frac{i}{2} (\gamma^0 \underline{\gamma^\nu \gamma^\mu} \gamma^0 - \gamma^0 \underline{\gamma^\mu \gamma^\nu} \gamma^0)$

$= -\frac{i}{2} \gamma^0 [\gamma^\nu, \gamma^\mu] \gamma^0$

$= \gamma^0 \sigma^{\mu\nu} \gamma^0$

5. Pokazati da je

a) $\gamma_5^\dagger = \gamma_5$, $\gamma^5 = \gamma_5^{-1}$

b) $(\gamma_5)^2 = 1$

c) $\{\gamma_5, \gamma_\mu\} = 0$

d) $(\gamma_5 \gamma_\mu)^\dagger = \gamma^0 \gamma_5 \gamma_\mu \gamma^0$

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a) Po definiciji

$$\gamma_5 = -i \gamma_0 \gamma_1 \gamma_2 \gamma_3$$

$$\gamma_5^\dagger = i \gamma_3^\dagger \gamma_2^\dagger \gamma_1^\dagger \gamma_0^\dagger$$

$$= i \gamma_0 \gamma_3 \underbrace{\gamma_0 \gamma_0}_{1} \gamma_2 \underbrace{\gamma_0 \gamma_0}_{1} \gamma_1 \underbrace{\gamma_0 \gamma_0 \gamma_0 \gamma_0}_{1}$$

$$= i \gamma_0 \gamma_3 \gamma_2 \gamma_1$$

$$= -i \gamma_0 \gamma_3 \gamma_1 \gamma_2 = i \gamma_0 \gamma_1 \gamma_3 \gamma_2$$

$$= -i \gamma_0 \gamma_1 \gamma_2 \gamma_3 = \gamma_5$$

$$\gamma^5 = \gamma_5^{-1}$$

$$\gamma_5^{-1} = (-i \gamma_0 \gamma_1 \gamma_2 \gamma_3)^{-1} = i \gamma_3^{-1} \gamma_2^{-1} \gamma_1^{-1} \gamma_0^{-1}$$

Potrebno da je

$$\{\gamma_\mu, \gamma^\nu\} = 2\delta^\nu_\mu$$

$$\gamma_0 \gamma^0 = 1 \Rightarrow \gamma_0^{-1} = \gamma^0$$

$$\gamma_i \gamma^i = 1 \Rightarrow \gamma_i^{-1} = \gamma^i$$

$$\gamma_5^{-1} = i\,\gamma^3\gamma^2\gamma^1\gamma^0$$

$$= -i\,\gamma^3\gamma^2\gamma^0\gamma^1 = i\,\gamma^3\gamma^0\gamma^2\gamma^1 =$$

$$= -i\,\gamma^0\gamma^3\gamma^2\gamma^1 = i\,\gamma^0\gamma^2\gamma^3\gamma^1 =$$

$$= -i\,\gamma^0\gamma^2\gamma^1\gamma^3 = i\,\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5$$

Sa druge strane:

$$\gamma^5 = i\,\gamma^0\gamma^1\gamma^2\gamma^3 = i\,\gamma_0 g^{00}\gamma_1 g^{11}\gamma_2 g^{22}\gamma_3 g^{33}$$

$$= i\,g^{00}g^{11}g^{22}g^{33}\,\gamma_0\gamma_1\gamma_2\gamma_3$$

$$= -i\,\gamma_0\gamma_1\gamma_2\gamma_3 = \gamma_5$$

Dakle

$$\gamma_5^{-1} = \gamma_5 \Rightarrow (\gamma_5)^2 = 1 \quad (b)$$

c) $\{\gamma_5, \gamma_\mu\} = 0$

↓

antikomutator

$$\gamma_0\gamma_5 = -i\,\gamma_0\gamma_0\gamma_1\gamma_2\gamma_3$$
$$= i\,\gamma_0\gamma_1\gamma_0\gamma_2\gamma_3$$
$$= -i\,\gamma_0\gamma_1\gamma_2\gamma_0\gamma_3$$
$$= i\,\gamma_0\gamma_1\gamma_2\gamma_3\gamma_0$$
$$= -(-i\,\gamma_0\gamma_1\gamma_2\gamma_3)\gamma_0$$
$$= -\gamma_5\gamma_0$$
$$\Downarrow$$
$$\{\gamma_0,\gamma_5\} = 0$$

$$\gamma_1\gamma_5 = -i\,\gamma_1\gamma_0\gamma_1\gamma_2\gamma_3$$
$$= i\,\gamma_0\gamma_1\gamma_1\gamma_2\gamma_3$$
$$= -i\,\gamma_0\gamma_1\gamma_2\gamma_1\gamma_3$$
$$= i\,\gamma_0\gamma_1\gamma_2\gamma_3\gamma_1$$
$$= -(-i\,\gamma_0\gamma_1\gamma_2\gamma_3)\gamma_1$$
$$= -\gamma_5\gamma_1$$

$$\{\gamma_1,\gamma_5\} = 0$$

Slično ovome $\{\gamma_2,\gamma_5\} = 0 = \{\gamma_3,\gamma_5\} \Rightarrow$

$$\Rightarrow \{\gamma_\mu,\gamma_5\} = 0$$

d) $(\gamma_5 \gamma_\mu)^\dagger = \gamma_\mu^\dagger \gamma_5^\dagger = \gamma_\mu^\dagger \gamma_5$

$= \gamma_0 \gamma_\mu \gamma_0 \gamma_5 = -\gamma_0 \gamma_\mu \gamma_5 \gamma_0 = \gamma_0 \gamma_5 \gamma_\mu \gamma_0$

$$\boxed{(\gamma_5 \gamma_\mu)^\dagger = \gamma_0 \gamma_5 \gamma_\mu \gamma_0}$$

6

Pokazati da je

$$\gamma_5 = -\frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$$

$$\epsilon_{\mu\nu\rho\sigma} = \begin{cases} +1 & \text{za parne permutacije od } 0123 \\ -1 & \text{za neparne per. od } 0123 \\ 0 & \text{ako su dva indeksa ista} \end{cases}$$

Permutacije od 0123

·0123	−1023	−2013	−3012
·0132	+1032	+2031	+3021
·0312	−1302	−2301	−3201
·0321	+1320	+2310	+3210
·0213	−1230	−2130	−3120
·0231	−1203	+2103	+3102

Ukupno 24 permutacija

Dakle na osnovi gornjih permutacija biće na primer

$$\gamma_5 = -\frac{i}{4!}\Big(\gamma^0\gamma^1\gamma^2\gamma^3 - \gamma^0\gamma^1\gamma^3\gamma^2 + \gamma^0\gamma^3\gamma^1\gamma^2$$

$$-\gamma^0\gamma^3\gamma^2\gamma^1 + \gamma^0\gamma^2\gamma^1\gamma^3 - \gamma^0\gamma^2\gamma^3\gamma^1 + \dots \text{itd}\Big)$$

Korišćenjem relacije

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

svih preostalih 23 člana (sem prvog) se mogu svesti na $\gamma^0\gamma^1\gamma^2\gamma^3$ i tako se stiže do jednakosti u postavci zadatka.

7. Pokazati da je

$$[\gamma_5, \sigma^{\mu\nu}] = 0$$

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$$

$$\left[\gamma_5, \frac{i}{2}[\gamma^\mu, \gamma^\nu]\right] = \frac{i}{2}[\gamma_5, \gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu]$$

$$= \frac{i}{2}\left(\gamma_5(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) - (\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\gamma_5\right)$$

$$= \frac{i}{2}\left(\gamma_5\gamma^\mu\gamma^\nu - \gamma_5\gamma^\nu\gamma^\mu - \gamma^\mu\gamma^\nu\gamma_5 + \gamma^\nu\gamma^\mu\gamma_5\right) =$$

$$\boxed{\{\gamma_5, \gamma^\mu\} = 0}$$

$$= \frac{i}{2}\left(-\gamma^\mu\gamma_5\gamma^\nu - \gamma_5\gamma^\nu\gamma^\mu - \gamma^\mu\gamma^\nu\gamma_5 - \gamma^\nu\gamma_5\gamma^\mu\right)$$

$$= \frac{i}{2}\left(\cancel{\gamma^\mu\gamma^\nu\gamma_5} \cancel{- \gamma_5\gamma^\nu\gamma^\mu} \cancel{- \gamma^\mu\gamma^\nu\gamma_5} + \cancel{\gamma_5\gamma^\nu\gamma^\nu}\right)$$

$$= 0$$

8

Pokazati da važi

a) $\gamma_\mu \gamma^\mu = 4$

b) $\gamma_\mu \gamma^\nu \gamma^\mu = -2\gamma^\nu$

c) $\gamma_\mu \gamma^\alpha \gamma^\beta \gamma^\mu = 4 g^{\alpha\beta}$

d) $\sigma_{\mu\nu} \sigma^{\mu\nu} = 12$

---

a) $\{\gamma_\mu, \gamma^\mu\} = 2\gamma_\mu \gamma^\mu = 2\delta^\mu_\mu = 8 \Rightarrow \gamma_\mu \gamma^\mu = 4$

↳ v. pozadi

b) $\gamma_\mu \gamma^\nu \gamma^\mu = -2\gamma^\nu$

$\gamma_\mu \gamma^\nu + \gamma^\nu \gamma_\mu = 2\delta^\mu_\nu$

$\gamma_\mu \gamma^\nu = -\gamma^\nu \gamma_\mu + 2\delta^\mu_\nu$

$\gamma_\mu \gamma^\nu \gamma^\mu = (-\gamma^\nu \gamma_\mu + 2\delta^\mu_\nu)\gamma^\mu$

$= -\gamma^\nu \gamma_\mu \gamma^\mu + 2\delta^\mu_\nu \gamma^\mu$

$= -4\gamma^\nu + 2\gamma^\nu = -2\gamma^\nu$

c) $\gamma_\mu \gamma^\alpha \gamma^\beta \gamma^\mu = 4 g^{\alpha\beta}$

$\gamma_\mu \gamma^\alpha \gamma^\beta \gamma^\mu = (-\gamma^\alpha \gamma_\mu \gamma^\beta \gamma^\mu + 2\delta^\alpha_\mu \gamma^\beta \gamma^\mu) =$

$$\gamma_\mu \gamma^\mu = \gamma_0\gamma^0 + \gamma_1\gamma^1 + \gamma_2\gamma^2 + \gamma_3\gamma^3$$

$$= \gamma^0\gamma^0 + g_{11}\gamma^1\gamma^1 + g_{22}\gamma^2\gamma^2 + g_{33}\gamma^3\gamma^3$$

$$= (\gamma^0)^2 - (\gamma^1)^2 - (\gamma^2)^2 - (\gamma^3)^2$$

$$= 1 - (-1) - (-1) - (-1)$$

$$= 4$$

$$= \left(-\gamma^\alpha \left(-\gamma^\beta \gamma_\mu + 2\delta^\beta_\mu \right) \gamma^\mu + 2\gamma^\beta \gamma^\alpha \right)$$

$$= \left(\gamma^\alpha \gamma^\beta \gamma_\mu \gamma^\mu - 2\delta^\beta_\mu \gamma^\alpha \gamma^\mu + 2\gamma^\beta \gamma^\alpha \right)$$

$$= \left(4\gamma^\alpha \gamma^\beta - 2\gamma^\alpha \gamma^\beta + 2\gamma^\beta \gamma^\alpha \right)$$

$$= 2\left(\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha \right) = 2\{\gamma^\alpha, \gamma^\beta\}$$

$$= 4 g^{\alpha\beta}$$

d) $\sigma_{\mu\nu} \sigma^{\mu\nu} = \frac{i}{2} [\gamma_\mu, \gamma_\nu] \frac{i}{2} [\gamma^\mu, \gamma^\nu]$

$$= -\frac{1}{4} \left((\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu)(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) \right)$$

$$= -\frac{1}{4} \left(\gamma_\mu \gamma_\nu \gamma^\mu \gamma^\nu - \gamma_\mu \gamma_\nu \gamma^\nu \gamma^\mu - \gamma_\nu \gamma_\mu \gamma^\mu \gamma^\nu \right.$$
$$\left. + \gamma_\nu \gamma_\mu \gamma^\nu \gamma^\mu \right)$$

$$= -\frac{1}{4} \left(\gamma_\mu \left(-\gamma^\mu \gamma_\nu + 2\delta^\mu_\nu \right) \gamma^\nu - 16 - \cancel{1}6 + \right.$$
$$\left. + \gamma_\nu \left(-\gamma^\nu \gamma_\mu + 2\delta^\nu_\mu \right) \gamma^\mu \right)$$

$$= -\frac{1}{4} \left(-\gamma_\mu \gamma^\mu \gamma_\nu \gamma^\nu + 2\delta^\mu_\nu \gamma_\mu \gamma^\nu - 32 + \right.$$
$$\left. - \gamma_\nu \gamma^\nu \gamma_\mu \gamma^\mu + 2\delta^\nu_\mu \gamma_\nu \gamma^\mu \right)$$

$$= -\frac{1}{4} \left(-16 + 8 - 32 - 16 + 8 \right) = 12$$

9. Pokazati da vati

a) $\mathrm{tr}\, \gamma_\mu = 0$

b) $\mathrm{tr}(\gamma_\mu \gamma_\nu) = 4 g_{\mu\nu}$

c) $\mathrm{tr}(\gamma_\mu \gamma_\nu \gamma_\varrho \gamma_\sigma) = 4(g_{\mu\nu} g_{\varrho\sigma} - g_{\mu\varrho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\varrho})$

d) $\mathrm{tr}\, \gamma_5 = 0$

~~e)~~ $\mathrm{tr}(\gamma_5 \gamma_\mu \gamma_\nu) = 0$

f) $\mathrm{tr}(\gamma_5 \gamma_\mu) = 0$

---

a) $\{\gamma_\mu, \gamma_5\} = 0$, $(\gamma^5)^2 = 1$; $(\gamma_5)^2 = 1$

$$\mathrm{tr}\, \gamma_\mu = \mathrm{tr}(\gamma_\mu \gamma_5 \gamma_5) = -\mathrm{tr}(\gamma_5 \gamma_\mu \gamma_5)$$

$$= -\mathrm{tr}(\gamma_5^2 \gamma_\mu) = -\mathrm{tr}(\gamma_\mu) \Rightarrow \mathrm{tr}\, \gamma_\mu = 0$$

b) $\mathrm{tr}(\gamma_\mu \gamma_\nu) = ?$

$$\{\gamma_\mu, \gamma_\nu\} = 2 g_{\mu\nu}$$

matrični Element

$$\mathrm{tr}(\gamma_\mu \gamma_\nu) = \mathrm{tr}(-\gamma_\nu \gamma_\mu + 2 g_{\mu\nu} I)$$

$$= -\mathrm{tr}(\gamma_\mu \gamma_\nu) + 2 g_{\mu\nu}\, \mathrm{tr}\, I \Rightarrow$$

$$2\, \mathrm{tr}(\gamma_\mu \gamma_\nu) = 2 g_{\mu\nu} \cdot 4$$

$$= 8 g_{\mu\nu} \Rightarrow \mathrm{tr}(\gamma_\mu \gamma_\nu) = 4 g_{\mu\nu}$$

c) $\mathrm{tr}\left[\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma\right] =$

$= \mathrm{tr}\left[\left(2g_{\mu\nu} - \gamma_\nu \gamma_\mu\right) \gamma_\rho \gamma_\sigma\right] = 2g_{\mu\nu}\, \mathrm{tr}\, \gamma_\rho \gamma_\sigma \; -$

$- \mathrm{tr}\, \gamma_\nu \gamma_\mu \gamma_\rho \gamma_\sigma = 2g_{\mu\nu}\, \mathrm{tr}\, \gamma_\rho \gamma_\sigma \; -$

$- \mathrm{tr}\, \gamma_\nu \left(2g_{\mu\rho} - \gamma_\rho \gamma_\mu\right) \gamma_\sigma = 2g_{\mu\nu}\, \mathrm{tr}\, \gamma_\rho \gamma_\sigma \; -$

$-2g_{\mu\rho}\, \mathrm{tr}\, \gamma_\nu \gamma_\sigma \quad + \mathrm{tr}\, \gamma_\nu \gamma_\rho \gamma_\mu \gamma_\sigma =$

$2g_{\mu\nu}\, \mathrm{tr}\, \gamma_\rho \gamma_\sigma - 2g_{\mu\rho}\, \mathrm{tr}\, \gamma_\nu \gamma_\sigma + \mathrm{tr}\, \gamma_\nu \gamma_\rho \left(2g_{\mu\sigma} - \gamma_\sigma \gamma_\mu\right.$

$2g_{\mu\nu} \cdot 4g_{\rho\sigma} - 2g_{\mu\rho} \cdot 4g_{\nu\sigma} + 2g_{\mu\sigma} \cdot 4g_{\nu\rho} \; -$

$- \mathrm{tr}\, \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_\mu$

Dalle

$\mathrm{tr}\, \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma = 8\left(g_{\mu\nu} g_{\rho\sigma} - g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\rho}\right)$

$- \mathrm{tr}\, \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \quad \Longrightarrow$

$\mathrm{tr}\, \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma = 4\left(g_{\mu\nu} g_{\rho\sigma} - g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\rho}\right)$

d) $\mathrm{tr}\, \gamma_5 = 0$

$$\mathrm{tr}\,\gamma_5 = \mathrm{tr}(\gamma_5\gamma_0\gamma_0) = -\mathrm{tr}(\gamma_0\gamma_5\gamma_0)$$

$$= -\mathrm{tr}(\gamma_0^2\gamma_5) = -\mathrm{tr}(\gamma_5) \Rightarrow \mathrm{tr}\,\gamma_5 = 0$$

e) $\mathrm{tr}(\gamma_5\gamma_\mu\gamma_\nu) = 0$

$\gamma^\alpha\gamma_\alpha = 4$ ⟵ Iz drugog zadatka

$$\mathrm{tr}(\gamma_5\gamma_\mu\gamma_\nu) = \frac{1}{4}\mathrm{tr}(\gamma_5\gamma^\alpha\gamma_\alpha\gamma_\mu\gamma_\nu)$$

$$= \frac{1}{4}\mathrm{tr}(\gamma_\alpha\gamma_\mu\gamma_\nu\gamma_5\gamma^\alpha) = -\frac{1}{4}\mathrm{tr}(\gamma_\alpha\gamma_\mu\gamma_\nu\gamma^\alpha\gamma_5)$$

$$= -\frac{1}{4}\mathrm{tr}(\gamma_5\underbrace{\gamma_\alpha\gamma_\mu\gamma_\nu\gamma^\alpha}_{4g_{\mu\nu}}) = -g_{\mu\nu}\,\mathrm{tr}(\gamma_5) = 0$$

Zadatak 8.

$\boxed{\text{Iskoriscena je i relacija } \{\gamma_5,\gamma^\mu\} = 0 \text{ iz Zad. 7.}}$

f) $\mathrm{tr}(\gamma_5\gamma_\mu) = 0$

$$\{\gamma_5, \gamma_\mu\} = 0$$

$$\mathrm{tr}\,\gamma_5\gamma_\mu = -\mathrm{tr}(\gamma_\mu\gamma_5) = -\mathrm{tr}(\gamma_5\gamma_\mu)$$

$$\Downarrow$$

$$\mathrm{tr}(\gamma_5\gamma_\mu) = 0$$

10. Izračunati

$$\gamma_\mu (1-\gamma_5)(\not{p}-m)\gamma^\mu$$

$$\{\gamma_\mu, \gamma_5\} = 0$$
$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$$

Prethodni zadaci

$$\gamma_\mu (1-\gamma_5)(\not{p}-m)\gamma^\mu =$$

$$\gamma_\mu (\not{p}-m-\gamma_5\not{p}+m\gamma_5)\gamma^\mu =$$

$$\gamma_\mu \not{p}\gamma^\mu - m\gamma_\mu\gamma^\mu - \gamma_\mu\gamma_5\not{p}\gamma^\mu + m\gamma_\mu\gamma_5\gamma^\mu =$$

$$p_\nu \gamma_\mu \gamma^\nu\gamma^\mu - 4m - p_\nu\gamma_\mu\gamma_5\gamma^\nu\gamma^\mu - m\gamma_5\underbrace{\gamma_\mu\gamma^\mu}_{4} =$$

$$p_\nu\gamma_\mu(-\gamma^\mu\gamma^\nu + 2g^{\mu\nu}) - 4m(1+\gamma_5) +$$

$$+p_\nu\gamma_5\gamma_\mu\gamma^\nu\gamma^\mu =$$

$$-4p_\nu\gamma^\nu + 2g^{\mu\nu}p_\nu\gamma_\mu - 4m(1+\gamma_5) + p_\nu\gamma_5\gamma_\mu(-\gamma^\mu\gamma^\nu + 2g$$

$$-4\not{p} + 2\not{p} - 4m(1+\gamma_5) + p_\nu\gamma_5\gamma_\mu\gamma^\mu\gamma^\nu + 2g^{\mu\nu}p_\nu\gamma_5\gamma_\mu =$$

$$-2\not{p} - 4m(1+\gamma_5) - 4\gamma_5\not{p} + 2\gamma_5\not{p} =$$

$$= -2\not{p} - 4m(1+\gamma_5) - 2\gamma_5\not{p}$$

$$= -2(1+\gamma_5)\not{p} - 4m(1+\gamma_5)$$

$$= -2(1+\gamma_5)(\not{p}+2m)$$

11. Pokazati da je

$$e^{\gamma_5 \not{a}} = \cos\sqrt{a_\mu a^\mu} + \frac{1}{\sqrt{a_\mu a^\mu}} \gamma_5 \not{a} \sin\sqrt{a_\mu a^\mu}$$

$$e^{\gamma_5 \not{a}} = 1 + \gamma_5 \not{a} + \frac{1}{2}(\gamma_5 \not{a})^2 + \frac{1}{3!}(\gamma_5 \not{a})^3 + \dots$$

$$(\gamma_5 \not{a})^2 = ?$$

$$(\gamma_5 \not{a})^2 = (\gamma_5 \gamma_\mu a^\mu)^2 = \gamma_5 \gamma_\mu a^\mu \gamma_5 \gamma_\nu a^\nu$$

$$= \gamma_5 \gamma_\mu \gamma_5 \gamma_\nu a^\mu a^\nu = -(\gamma_5)^2 \gamma_\mu \gamma_\nu a^\mu a^\nu$$

$$\{\gamma_5, \gamma_\mu\} = 0$$

$$= -(\gamma_\mu \gamma_\nu) a^\mu a^\nu = -\not{a}^2$$

$$\not{a}^2 = \gamma_\mu \gamma_\nu a^\mu a^\nu = (-\gamma_\nu \gamma_\mu + 2g_{\mu\nu}) a^\mu a^\nu$$

$$= -\gamma_\nu \gamma_\mu a^\mu a^\nu + 2g_{\mu\nu} a^\mu a^\nu$$

$$= -\not{a}^2 + 2g_{\mu\nu} a^\mu a^\nu$$

$$\not{a}^2 = -\not{a}^2 + 2a^2$$

$$\Downarrow$$

$$\not{a}^2 = a^2$$

$$(\gamma_5 \not{a})^2 = -a^2$$

$$(\gamma_5 \not{a})^3 = (\gamma_5 \not{a})^2 (\gamma_5 \not{a}) = -a^2 (\gamma_5 \not{a})$$

$$(\gamma_5 \not{a})^4 = (\gamma_5 \not{a})^2 (\gamma_5 \not{a})^2 = a^4$$

$$(\gamma_5 \not{a})^5 = (\gamma_5 \not{a})^4 (\gamma_5 \not{a}) = a^4 (\gamma_5 \not{a})$$

$$\vdots$$

$$e^{\gamma_5 \not{a}} = 1 + \gamma_5 \not{a} - \frac{1}{2} a^2 - \frac{a^2}{3!} (\gamma_5 \not{a}) +$$

$$\frac{1}{4!} a^4 + \frac{a^4}{5!} (\gamma_5 \not{a}) + \ldots$$

$$= \left(1 - \frac{1}{2!} a^2 + \frac{a^4}{4!} + \ldots\right) + (\gamma_5 \not{a}) \frac{a}{a} \left(1 - \frac{a^2}{3!} + \frac{a^4}{5!} + \ldots\right.$$

$$= \cos \sqrt{a^2} + \frac{1}{\sqrt{a^2}} (\gamma_5 \not{a}) \sin \sqrt{a^2}$$

$a^2 = a_\mu a^\mu$

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} + \ldots$$

$$\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right)$$

$e^{ix} = \cos x + i \sin x$ (Euler)

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \qquad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$$

12.

Neka je dat skup matrica

$$\Gamma^a = \{ I, \gamma^\mu, \gamma^5, \gamma^\mu\gamma^5, \sigma^{\mu\nu} \}$$

sa osobinama

1. $(\Gamma^a)^2 = I$
2. $\Gamma^a\Gamma^b = \alpha\,\Gamma^c \quad \begin{cases} a=b, & \alpha=1 \\ a\neq b, & \pm 1, \pm i \end{cases}$
3. Ako je $\Gamma^a \neq I$, onda uvek postoji neko Γ^b takvo da važi

$$\Gamma^b\Gamma^a\Gamma^b = -\Gamma^a \iff \{\Gamma^b, \Gamma^a\} = 0$$

4. Ako je $\Gamma^a \neq I$

$$\mathrm{tr}\,\Gamma^a = 0 \quad , \quad a = 2, \dots 16$$

Pokazati da Γ^a-ovi čine LNZ skup i da se proizvoljna matrica $A \in C^{44}$ može zapisati kao

$$A = \sum_a \theta_a \Gamma^a$$

gde je $\theta_a = \frac{1}{4}\,\mathrm{tr}(A\Gamma^a)$

LNZ скупа Γ^a

$$\sum_{a=1}^{16} \beta_a \Gamma^a = 0$$

$$\beta_a = 0 \,, \forall a \iff \Gamma^a \ \text{LNZ}$$

$$tr \sum_{a=1}^{16} \beta_a \Gamma^a = 0$$

$$tr \beta_1 \Gamma^1 + tr \sum_{a=2}^{16} \beta_a \Gamma^a = 0$$

$$4\beta_1 + \sum_{a=2}^{16} \beta_a \underbrace{\cancel{tr \Gamma^a}}_{0} = 0$$

$$\boxed{\beta_1 = 0}$$

$$\sum_{a=2}^{16} \beta_a \Gamma^a = 0 \quad / \Gamma^b$$

$$\sum_{a=2}^{16} \beta_a \Gamma^a \Gamma^b = 0$$

$$\beta_a \underbrace{(\Gamma^a)^2}_{1} + \sum_{a \neq b} \beta_a \Gamma^a \Gamma^b = 0$$

$$\beta_a + \sum_{a \neq b} \beta_a \Gamma^a \Gamma^b = 0 \quad / tr$$

Матрица (јединична) 4x4

$$4\beta_a + \sum_{a\neq b} \beta_a \, tr\left(\Gamma^a\Gamma^b\right) = 0$$

3. $$\{\Gamma^b, \Gamma^a\} = 0 \Rightarrow tr\, \Gamma^a\Gamma^b = 0$$

$$\beta_a = 0, \quad a = 2, \dots 16$$

$$A = \sum_a \theta_a \Gamma^a$$

$$A\Gamma^b = \sum_a \theta_a \Gamma^a \Gamma^b$$

$$= \theta_b I + \sum_{a\neq b} \theta_a \Gamma^a\Gamma^b$$

$$tr\, A\Gamma^b = 4\theta_b + \sum_{a\neq b} \theta_a \, \cancel{tr\left(\Gamma^a\Gamma^b\right)}^{\,0}$$

$$\boxed{\theta_b = \frac{1}{4}\, tr A\Gamma^b}$$

13. Pokazati da je za Dirakovu j-nu četvorovektor struje dat izrazom

$$j^\mu = \bar{\psi} \gamma^\mu \psi$$

gde je $\bar{\psi} = \psi^+ \gamma^0$

$\bar{\psi}$ – adjoint

(ψ надвучено, српски)

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \qquad (1)$$

$$(\gamma^\mu)^+ = \gamma^0 \gamma^\mu \gamma^0 \Rightarrow (\gamma^\mu)^+ \gamma^0 \overset{(\gamma^0)^2=1}{=} \gamma^0 \gamma^\mu$$

Iz $\boxed{(1)}$, ~~kompleksnom~~ ~~konjugacijom~~ (adjungovanjem) $\Rightarrow$

$$\psi^+ (-i(\gamma^\mu)^+ \partial_\mu - m) = 0 \quad / \gamma^0$$

$$\psi^+ (-i(\gamma^\mu)^+ \gamma^0 \partial_\mu - m\gamma^0) = 0$$

$$\psi^+ \gamma^0 (-i\gamma^\mu \partial_\mu - m) = 0$$

$$\bar{\psi} (-i\gamma^\mu \partial_\mu - m) = 0$$

$$\bar{\psi} (i\gamma^\mu \overleftarrow{\partial_\mu} + m) = 0 \qquad (2)$$

Pomnožiti poslednju j-nu sa ψ s desna a (1) sa $\bar{\psi}$ s leva:

$$\bar{\Psi}(-i\gamma^\mu \overleftarrow{\partial}_\mu - m)\Psi = 0$$

$$\bar{\Psi}(i\gamma^\mu \overrightarrow{\partial}_\mu - m)\Psi = 0$$

$\Big\} \Rightarrow$

$$i\bar{\Psi}\gamma^\mu \overleftarrow{\partial}_\mu \Psi = -m\bar{\Psi}\Psi$$

$$i\bar{\Psi}\gamma^\mu \overrightarrow{\partial}_\mu \Psi = m\bar{\Psi}\Psi$$

$\bigoplus$

$$i\,\partial_\mu(\bar{\Psi}\gamma^\mu\Psi) = 0$$

$\Downarrow$ $\partial_\mu j^\mu = 0$ – jednačina kontinuiteta

$$\boxed{j^\mu = \bar{\Psi}\gamma^\mu\Psi}$$

14. Rešiti Dirakovu j-nu za slobodnu česticu, pretpostavljajući da Hamiltonijan ne zavisi eksplicitno od vremena.

Dirakova j-na

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0 \quad ; \quad x = x^\mu$$

$\not\partial$

$\psi(x)$ – četvorokomponentni spinor

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix}$$

Hamiltonijan ne zavisi eksplicitno od vremena pa se rešenje može tražiti u obliku

$$\psi(x) = \psi(\vec{r}, t) = \phi(\vec{r})\, e^{-\frac{i}{\hbar}Et} \qquad (1)$$

što znači da se Dirakova j-na svodi na

$$\hat{H}_D \phi(\vec{r}) = E\, \phi(\vec{r}) \qquad \text{(stacionarna j-na)}$$

uveriti se

Budući da je zadovoljeno $[H_D, \vec{P}] = 0$, po analogiji sa kvantnom mehanikom sledi

$$\phi_p(\vec{r}) = u_p\, e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}} \qquad (2)$$

odnosno

$$\psi(x) = u_p\, e^{\frac{i}{\hbar}\vec{p}\vec{r}}\, e^{-\frac{i}{\hbar}Et} = u_p\, e^{\frac{i}{\hbar}(\vec{r}\cdot\vec{p} - Et)}$$

U_p – konstantan spinor

Ovo partikularno rešenje mora da zadovolji Dirakovu j-nu

$$(\not p - m)\,\psi(x) = 0$$

odnosno

$$(\not p - m)\,U_p = 0 \qquad (3)$$

U_p zapisujemo u obliku

$$U_p = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}_{4\times 1} \qquad (4)$$

$$\not p = p_\mu \gamma^\mu = p_0 \gamma^0 - \vec{p}\cdot\vec{\gamma}$$

$$= p_0 \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} - \vec{p} \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix} - \begin{pmatrix} 0 & \vec{p}\cdot\vec{\sigma} \\ -\vec{p}\cdot\vec{\sigma} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} E & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -E \end{pmatrix}_{4\times 4}$$

↓

БЛОК-МАТРИЦА

Koristeći (3) i (4), sledi

$$\left[\begin{pmatrix} E & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -E \end{pmatrix} - \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}_{4\times4}\right]\begin{pmatrix} \varphi \\ \chi \end{pmatrix}_{4\times1} = 0$$

$$\begin{pmatrix} E-m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -E-m \end{pmatrix}\begin{pmatrix} \varphi \\ \chi \end{pmatrix} = 0 \qquad (5)$$

Homogeni sistem (5) ima rešenje ako je determinanta odgovarajuće matrice nula

$$-(E^2-m^2) + (\vec{\sigma}\cdot\vec{p})^2 = 0$$

$$-E^2+m^2 + \vec{p}\cdot\vec{p} + i\,\vec{\sigma}(\vec{p}\times\vec{p}) = 0$$

$$-E^2 = -m^2-p^2$$

[Kvantna mehanika

$$(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b} + i\,\vec{\sigma}\cdot(\vec{a}\times\vec{b})$$

$$E^2 = m^2+p^2$$

$$E = \pm\sqrt{p^2+m^2} = \pm E_p$$

Iz (5) $\Rightarrow$

$$(E-m)\varphi - \vec{\sigma}\cdot\vec{p}\,\chi = 0 \qquad (6) \quad \text{za } E=-E_p$$

$$\vec{\sigma}\cdot\vec{p}\,\varphi - (E+m)\chi = 0 \qquad (7) \quad \text{za } E=E_p$$

Iz (7), $E = E_p$

$$\chi = \frac{\vec{\sigma}\cdot\vec{p}}{E_p + m}\varphi$$

Spinor onda glasi

$$U_p = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = \begin{pmatrix} \varphi \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\varphi \end{pmatrix} \equiv U(E_p, \vec{p})$$

Iz (6), $E \equiv -E_p$

$$\varphi = -\frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\chi$$

Spinor onda glasi

$$U_p = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = \begin{pmatrix} -\frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\chi \\ \chi \end{pmatrix} \equiv U(-E_p, \vec{p})$$

Ako se uvedu nove oznake

$$v(\vec{p}) = U(-E_p, -\vec{p}) = \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\chi \\ \chi \end{pmatrix}$$

i $u(\vec{p}) = u(E_p, \vec{p}) = \begin{pmatrix} \varphi \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\varphi \end{pmatrix}$

LNZ rešenja Diracove jne glase

$u(\vec{p})e^{-ip\cdot x}$ i $v(\vec{p})e^{ip\cdot x}$

32 ...

Iz prethodnog zadatka je poznato

$$j^\mu = \bar{\psi}\gamma^\mu\psi$$

odnosno

$$j^0 = \rho = \psi^+ \underbrace{\gamma^0\gamma^0}_{1} \psi = \psi^+\psi \quad (\overset{\text{аналогија}}{\rho = |\psi|^2})$$

Posmatrajmo, formalno, ρ kao statistički operator u prostoru Minkovskog, pri čemu važi

$$\mathrm{tr}\,\rho = 1$$

Na osnovi ovoga se mogu normalizovati dobijena LNZ rešenja.

$$\psi = N\,U(\vec{p})\,e^{-ip\cdot x}$$

$$\rho = |N|^2 U^+(\vec{p})\,U(\vec{p})$$

$$= |N|^2 \left(\varphi^+ \quad \varphi^+ \frac{(\vec{\sigma}\cdot\vec{p})^+}{E_p+m}\right)\begin{pmatrix}\varphi \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\varphi\end{pmatrix}$$

$$= |N|^2 \varphi^+ \left(1 \quad \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\right)\begin{pmatrix}1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\end{pmatrix}\varphi \quad ; \quad \varphi = \begin{bmatrix}1\\0\end{bmatrix}$$

избор

$$= |N|^2 \left(1 + \frac{(\vec{\sigma}\cdot\vec{p})^2}{(E_p+m)^2}\right) \underbrace{\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}}_{1}$$

Već je nađeno $(\vec{\sigma}\cdot\vec{p})^2 = E_p^2 - m^2$

$$\rho = |N|^2 \left(1 + \frac{E_p^2 - m^2}{(E_p+m)^2}\right) =$$

$$= |N|^2 \left(1 + \frac{E_p - m}{E_p + m}\right) = |N|^2 \, \frac{E_p + \cancel{m} + E_p - \cancel{m}}{E_p + m}$$

$$= |N|^2 \frac{2E_p}{E_p + m}$$

$$\mathrm{tr}\,\rho = 1 \quad \Rightarrow \quad N = \sqrt{\frac{E_p+m}{2E_p}}\, e^{i\delta}.$$

Dakle, jedno rešenje je

$$\psi(x) = \sqrt{\frac{E_p+m}{2E_p}}\, u(\vec{p})\, e^{-ip\cdot x}$$

$$= \frac{1}{\sqrt{2E_p}}\, u(\vec{p})\, e^{-ip\cdot x}$$

Opšte rešenje

$$\psi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \sum_{r=1}^{2} \int d^3\vec{p} \left(c_r(\vec{p})\, \psi_r^+(x) + d_r^+(\vec{p})\, \psi_r^-(x) \right)$$

$$= \frac{1}{(2\pi)^{3/2}} \sum_{r=1}^{2} \int \frac{d^3\vec{p}}{\sqrt{2E_p}} \left(c_r(\vec{p})\, u_r(\vec{p})\, e^{-ipx} + d_r^+(\vec{p})\, v_r(\vec{p})\, e^{ipx} \right)$$

35. Pokazati kako izgleda rešenje Dirakove j-ne za $E > 0$ u nerelativističkom limesu.

$$\Psi = N \begin{pmatrix} \varphi \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p + m}\varphi \end{pmatrix} e^{-ip\cdot x}$$

$$\chi = \frac{\vec{\sigma}\cdot\vec{p}}{E_p + m}\varphi$$

$$E_p^2 = \vec{p}^2 c^2 + m_0^2 c^4$$

$$= m_0^2 c^4 \left(1 + \frac{\vec{p}^2}{m_0^2 c^2}\right)$$

$$E_p = m_0 c^2 \sqrt{1 + \frac{\vec{p}^2}{m_0^2 c^2}}$$

$$\left[(1+x)^{\frac{1}{2}} \approx 1 + \frac{1}{2}x, \quad x \ll 1 \right]$$

$$\approx m_0 c^2 \left(1 + \frac{1}{2}\frac{\vec{p}^2}{m_0^2 c^2}\right)$$

$c = 1$

$$\approx m_0 + \frac{\vec{p}^2}{2m_0} \Rightarrow \boxed{E_p - m_0 = \frac{\vec{p}^2}{2m_0}}$$

$$\Downarrow$$

$$E_p + m_0 - m_0 - m_0 = \frac{\vec{p}^2}{2m_0}$$

$$\boxed{E_p + m_0 = 2m_0 + \frac{\vec{p}^2}{2m_0}}$$

$$m_0 \gg \frac{p^2}{2m_0}$$

$$\chi = \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m_0}\varphi \quad \Rightarrow \quad \left(\vec{\sigma}\cdot\vec{p} = \sqrt{E_p^2 - m_0^2} \right)$$

$$\frac{\chi}{\varphi} = \frac{\sqrt{E_p^2-m_0^2}}{E_p+m_0} = \sqrt{\frac{E_p^2-m_0^2}{(E_p+m_0)^2}} = \sqrt{\frac{E_p-m_0}{E_p+m_0}} \simeq$$

$$= \sqrt{\frac{\frac{p^2}{2m_0}}{2m_0+\frac{p^2}{2m_0}}} = \sqrt{\frac{1}{\frac{2m_0}{\frac{p^2}{2m_0}}+1}} = \sqrt{\frac{1}{\frac{4m_0^2}{p^2}+1}} \ll 1$$

Onda je

$$\Psi = N \begin{pmatrix} \varphi \\ 0 \end{pmatrix} e^{-ipx}$$

$$\varphi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ ili } \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

pa će biti

$$\left.\begin{aligned} \Psi &= N \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-ipx} \quad \text{ili} \\ \Psi &= N \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{-ipx} \end{aligned}\right\} \text{ rešenje u nerelativističkom limesu}$$

110. Pokazati važenje relacija

$$\sum_{r=1}^{2} u_r(\vec{p})\, \bar{u}_r(\vec{p}) = \not{p} + m$$

i
$$\sum_{r=1}^{2} v_r(\vec{p})\, \bar{v}_r(\vec{p}) = \not{p} - m$$

$$u_r(\vec{p}) = \sqrt{E_p + m} \begin{pmatrix} \varphi_r \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\varphi_r \end{pmatrix}$$

$$\sum_{r=1}^{2} u_r(\vec{p})\, \bar{u}_r(\vec{p}) = \sum_{r=1}^{2} u_r(\vec{p})\, u_r^{+}(\vec{p})\,\gamma^0 =$$

$$= u_1(\vec{p})\, u_1^{+}(\vec{p})\,\gamma^0 + u_2(\vec{p})\, u_2^{+}(\vec{p})\,\gamma^0$$

$$= (E_p+m)\left[\begin{pmatrix} \varphi_1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\varphi_1 \end{pmatrix} \begin{pmatrix} \varphi_1^{+} & \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\varphi_1^{+} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} + \right.$$

$$\left. \begin{pmatrix} \varphi_2 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\varphi_2 \end{pmatrix} \begin{pmatrix} \varphi_2^{+} & \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\varphi_2^{+} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \right] =$$

$$\varphi_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \varphi_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

⋮

$$\begin{pmatrix} E_p+m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -(E_p+m) \end{pmatrix} = \not{p} + m$$ ✓

(V. zadatak 14)

Drugo za domaći!